

Spectral Density of Complex Networks with Two Species of Nodes

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Abstract

The adjacency and Laplacian matrices of complex networks with two species of nodes are studied and the spectral density is evaluated by using the replica method in statistical physics. The network nodes are classified into two species (A and B) and the connections are made only between the nodes of different species. A static model of such networks with power law degree distributions is introduced by applying Goh, Kahng and Kim's method to construct scale free networks. As a result, the spectral density is shown to obey a power law in the limit of large mean degree.

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1 Introduction

The theory of complex networks, which has dramatically been developed since the end of the last century, is based on the observation that there are universal features in real biological and social networks[1]. One of such features is the scale free property, meaning that the degree (the number of nodes directly connected to each node) distribution function $P(\Delta)$ obeys a power law $P(\Delta) \propto \Delta^{-\lambda}$ for large Δ . Barabási and Albert explained the origin of this scale free property by focusing on the network growing process[2]. Goh, Kahng and Kim formulated a static network model which exhibits the scale free property[3].

The connection pattern of a network is mathematically described by the adjacency matrix. When the network has the scale free property, the spectral (eigenvalue) density $\rho(\mu)$ of the adjacency matrix is also expected to obey a power law $\rho(\mu) \propto \mu^{-\gamma}$ for large μ . Dorogovtsev et al. presented an analytic evidence of this power law behaviour[4, 5]. Moreover a relation $\gamma = 2\lambda - 1$ was found between the exponents of the power laws. Rodgers et al. analysed Goh, Kahng and Kim's static model and confirmed the power law behaviour of $\rho(\mu)$ [6].

In this paper, we shall study scale free networks with two species (A and B) of nodes. The connections are made only between the nodes of different species. We introduce a static model of such scale free networks by applying Goh, Kahng and Kim's method, and observe that each species has its own degree distribution function obeying a power law. Suppose that the exponent of the degree distribution function is λ_A for the species A and λ_B for the species B. Using the replica method in statistical physics, we are able to analytically evaluate the spectral density $\rho(\mu)$ in the limit of large mean degree[6, 7, 8]. As a result, we find that $\rho(\mu)$ also obeys a power law and the exponent γ is associated with the exponents λ_A and λ_B . In addition, the spectral density of the Laplacian matrix is similarly analysed and the power law behaviour is confirmed.

The outline of this paper is as follows. In §2, a static model of scale free networks with two species of nodes is introduced, and the adjacency and Laplacian matrices are defined. In §3, in order to evaluate the spectral density, we apply the replica method to the network model. In §4, in the limit of large mean degree, the power law behaviour of the spectral density is analytically derived. In §4, the effective medium approximation is briefly discussed as an attempt to treat the case with a finite mean degree.

2 Complex Networks with Two Species of Nodes

Let us suppose that there are N nodes of type A and M nodes of type B ($N \geq M$). We are interested in the asymptotic behaviour of networks with two species of nodes A and B in the limit

$$N \rightarrow \infty \text{ and } M \rightarrow \infty \text{ with } c = N/M \text{ fixed.} \quad (2.1)$$

We introduce a static model of such networks with power law degree distributions by applying Goh, Kahng and Kim's method. Each node of type A is assigned a probability P_j normalised as

$$\sum_{j=1}^N P_j = 1, \quad (2.2)$$

while each node of type B has a probability Q_k with

$$\sum_{k=1}^M Q_k = 1. \quad (2.3)$$

The nodes of type A and B are connected according to the following procedure. In each step we choose a node j of type A and a node k of type B with probabilities P_j and Q_k , respectively. Then the nodes j and k are connected, unless they are already connected. After repeating such a step pN times, a node j of type A and a node k of type B is connected with a probability

$$f_{jk} = 1 - (1 - P_j Q_k)^{pN} \sim 1 - e^{-pN P_j Q_k}. \quad (2.4)$$

Let us consider an $N \times M$ matrix C ($N \geq M$), where $C_{jk} = 1$ if the node j of type A is directly connected to the node k of type B, and $C_{jk} = 0$ otherwise. This random matrix C describes the connection pattern of the network with two species of nodes. Each matrix element C_{jk} is independently distributed with the probability density function (p.d.f.)

$$\mathcal{P}_{jk}(C_{jk}) = (1 - f_{jk})\delta(C_{jk}) + f_{jk}\delta(1 - C_{jk}). \quad (2.5)$$

We assume that P_j and Q_k are given by

$$P_j = \frac{j^{-\alpha}}{\sum_{l=1}^N l^{-\alpha}} \sim (1 - \alpha)N^{\alpha-1}j^{-\alpha}, \quad 0 < \alpha < 1 \quad (2.6)$$

and

$$Q_k = \frac{k^{-\beta}}{\sum_{l=1}^M l^{-\beta}} \sim (1-\beta)M^{\beta-1}k^{-\beta}, \quad 0 < \beta < 1. \quad (2.7)$$

There are thus two parameters α and β controlling the p.d.f. of the matrix C .

We define the degree d_j of the type-A node j as the number of directly connected type-B nodes:

$$d_j = \sum_{k=1}^M C_{jk}. \quad (2.8)$$

Then the type-A node degree distribution function is given by

$$P^{(A)}(\Delta) = \left\langle \frac{1}{N} \sum_{j=1}^N \delta(\Delta - d_j) \right\rangle, \quad (2.9)$$

where the brackets denote the average over the p.d.f. (2.5) and $\delta(x)$ is Dirac's delta function. We can similarly introduce the degree e_k of the type-B node k :

$$e_k = \sum_{j=1}^N C_{jk} \quad (2.10)$$

and the type-B node degree distribution function

$$P^{(B)}(\Delta) = \left\langle \frac{1}{M} \sum_{k=1}^M \delta(\Delta - e_k) \right\rangle. \quad (2.11)$$

In Appendix, a useful asymptotic relation

$$\ln \left\langle \exp \left(-i \sum_{j=1}^N \sum_{k=1}^M C_{jk} t_{jk} \right) \right\rangle \sim pN \sum_{j=1}^N \sum_{k=1}^M P_j Q_k (e^{-it_{jk}} - 1) \quad (2.12)$$

is derived in the limit (2.1). Here t_{jk} , which depends on neither N nor M , is in the neighbourhood of the origin so that $|e^{-it_{jk}} - 1| < 1$.

As special cases, we can readily derive asymptotic relations for

$$F_j^{(A)}(t) = \ln \langle e^{-id_j t} \rangle, \quad F_k^{(B)}(t) = \ln \langle e^{-ie_k t} \rangle \quad (2.13)$$

as

$$F_j^{(A)}(t) \sim pN P_j (e^{-it} - 1), \quad F_k^{(B)}(t) \sim pN Q_k (e^{-it} - 1). \quad (2.14)$$

Then we can readily see that

$$\begin{aligned}\langle d_j \rangle &= i \left. \frac{\partial}{\partial t} F_j^{(A)}(t) \right|_{t=0} \sim p N P_j, \\ \langle e_k \rangle &= i \left. \frac{\partial}{\partial t} F_k^{(B)}(t) \right|_{t=0} \sim p N Q_k,\end{aligned}\tag{2.15}$$

so that the mean degree $m^{(A)}$ of the type-A node is

$$m^{(A)} = \frac{1}{N} \sum_{j=1}^N \langle d_j \rangle \sim p,\tag{2.16}$$

while the mean degree $m^{(B)}$ of the type-B node is

$$m^{(B)} = \frac{1}{M} \sum_{k=1}^M \langle e_k \rangle \sim p c.\tag{2.17}$$

It can be seen from (2.6), (2.9) and (2.14) that the type-A node degree distribution function can be written as

$$\begin{aligned}P^{(A)}(\Delta) &= \frac{1}{2\pi N} \sum_{j=1}^N \int dt e^{i\Delta t + F_j^{(A)}(t)} \\ &\sim \frac{1}{2\pi} \int dt \int_0^1 dx \exp \left\{ i\Delta t + p(1-\alpha)x^{-\alpha}(e^{-it} - 1) \right\}.\end{aligned}\tag{2.18}$$

Then in the limit $\Delta \rightarrow \infty$ we find

$$P^{(A)}(\Delta) \sim \int_0^1 dx \delta \left\{ \Delta - p(1-\alpha)x^{-\alpha} \right\} = \frac{\{p(1-\alpha)\}^{1/\alpha}}{\alpha} \frac{1}{\Delta^{(1/\alpha)+1}},\tag{2.19}$$

and similarly obtain

$$P^{(B)}(\Delta) \sim \int_0^1 dy \delta \left\{ \Delta - pc(1-\beta)y^{-\beta} \right\} = \frac{\{pc(1-\beta)\}^{1/\beta}}{\beta} \frac{1}{\Delta^{(1/\beta)+1}}.\tag{2.20}$$

Thus we have seen that the network has the scale free property, as the node degree distribution functions obey power laws. The exponents of the power laws defined as

$$P^{(A)}(\Delta) \propto \Delta^{-\lambda_A}, \quad P^{(B)}(\Delta) \propto \Delta^{-\lambda_B}, \quad \Delta \rightarrow \infty\tag{2.21}$$

are found to be $\lambda_A = (1/\alpha) + 1$ and $\lambda_B = (1/\beta) + 1$.

In this paper we study the adjacency and Laplacian matrices of this scale free network. The adjacency matrix \mathcal{A} of this network is defined as

$$\mathcal{A} = \begin{pmatrix} O_N & C \\ C^T & O_M \end{pmatrix}, \quad (2.22)$$

where C^T is a transpose of C and O_n is an $n \times n$ matrix with zero elements. The Laplacian matrix \mathcal{L} is an $(N + M) \times (N + M)$ symmetric matrix with

$$\mathcal{L}_{jl} = \begin{cases} d_j, & j = l \text{ and } 1 \leq j \leq N, \\ e_{j-N}, & j = l \text{ and } N + 1 \leq j \leq N + M, \\ -\mathcal{A}_{jl}, & j \neq l. \end{cases} \quad (2.23)$$

Let us define that J is the adjacency matrix \mathcal{A} or the Laplacian matrix \mathcal{L} . The spectral density of J is defined as

$$\rho(\mu) = \left\langle \frac{1}{N + M} \sum_{j=1}^{N+M} \delta(\mu - \mu_j) \right\rangle, \quad (2.24)$$

where μ_j , $j = 1, 2, \dots, N + M$ are the eigenvalues of J . In order to calculate $\rho(\mu)$, we introduce the partition function

$$Z(\mu) = \int \prod_{j=1}^{N+M} d\Phi_j \exp \left(\frac{i}{2} \mu \sum_{j=1}^{N+M} \Phi_j^2 - \frac{i}{2} \sum_{j=1}^{N+M} \sum_{l=1}^{N+M} J_{jl} \Phi_j \Phi_l \right). \quad (2.25)$$

Using the partition function Z , we can write the spectral density as

$$\begin{aligned} \rho(\mu) &= \frac{1}{(N + M)\pi} \text{ImTr} \left\langle \{J - (\mu + i\epsilon)I\}^{-1} \right\rangle \\ &= \frac{2}{(N + M)\pi} \text{Im} \frac{\partial}{\partial \mu} \langle \ln Z(\mu + i\epsilon) \rangle, \end{aligned} \quad (2.26)$$

where ϵ is an infinitesimal positive number and I is an $(N + M) \times (N + M)$ identity matrix. Then we can utilise the relation

$$\lim_{n \rightarrow 0} \frac{\ln \langle Z^n \rangle}{n} = \langle \ln Z \rangle \quad (2.27)$$

to obtain

$$\rho(\mu) = \lim_{n \rightarrow 0} \frac{2}{(N + M)n\pi} \text{Im} \frac{\partial}{\partial \mu} \ln \langle \{Z(\mu)\}^n \rangle. \quad (2.28)$$

Therefore it is necessary to evaluate the average $\langle Z^n \rangle$. The replica method explained below is known to be a powerful tool for that purpose.

3 Replica Method

Let us first discuss the spectral density of the adjacency matrix \mathcal{A} . The eigenvalues μ_j , $j = 1, 2, \dots, N + M$ of \mathcal{A} consist of M pairs $\pm\nu_j$, $j = 1, 2, \dots, M$ and $N - M$ zeros. Note that $\nu_j > 0$ are identified with the eigenvalues of the $M \times M$ correlation matrix V with

$$V_{kl} = \sum_{j=1}^N C_{jk} C_{jl}. \quad (3.1)$$

Using the notations

$$\phi_j = \Phi_j, \quad j = 1, 2, \dots, N \quad (3.2)$$

and

$$\psi_k = \Phi_{k+N}, \quad k = 1, 2, \dots, M, \quad (3.3)$$

we can rewrite the partition function Z as

$$Z(\mu) = \int \prod_{j=1}^N d\phi_j \int \prod_{k=1}^M d\psi_k \exp \left(\frac{i}{2} \mu \sum_{j=1}^N \phi_j^2 + \frac{i}{2} \mu \sum_{k=1}^M \psi_k^2 - i \sum_{j=1}^N \sum_{k=1}^M C_{jk} \phi_j \psi_k \right). \quad (3.4)$$

Then we introduce the replica variables

$$\vec{\phi}_j = (\phi_j^{(1)}, \phi_j^{(2)}, \dots, \phi_j^{(n)}), \quad \vec{\psi}_k = (\psi_k^{(1)}, \psi_k^{(2)}, \dots, \psi_k^{(n)}) \quad (3.5)$$

and

$$d\vec{\phi}_j = d\phi_j^{(1)} d\phi_j^{(2)} \dots d\phi_j^{(n)}, \quad d\vec{\psi}_k = d\psi_k^{(1)} d\psi_k^{(2)} \dots d\psi_k^{(n)} \quad (3.6)$$

to obtain

$$\begin{aligned} \langle Z^n \rangle &= \int \prod_{j=1}^N d\vec{\phi}_j \int \prod_{k=1}^M d\vec{\psi}_k \exp \left(\frac{i}{2} \mu \sum_{j=1}^N \vec{\phi}_j^2 + \frac{i}{2} \mu \sum_{k=1}^M \vec{\psi}_k^2 \right) \\ &\quad \times \left\langle \exp \left(-i \sum_{j=1}^N \sum_{k=1}^M C_{jk} \vec{\phi}_j \cdot \vec{\psi}_k \right) \right\rangle. \end{aligned} \quad (3.7)$$

Now we can see from (2.12) that

$$\left\langle \exp \left(-i \sum_{j=1}^N \sum_{k=1}^M C_{jk} \vec{\phi}_j \cdot \vec{\psi}_k \right) \right\rangle \sim \exp \left\{ pN \sum_{j=1}^N \sum_{k=1}^M P_j Q_k \left(e^{-i\vec{\phi}_j \cdot \vec{\psi}_k} - 1 \right) \right\}. \quad (3.8)$$

It should be noted that this asymptotic relation holds if $\vec{\phi}_j \cdot \vec{\psi}_k$ is in the neighbourhood of the origin. This condition is justified in the limit of large mean degree $p \rightarrow \infty$, since $\vec{\phi}_j^2$ and $\vec{\psi}_k^2$ are scaled as $O(p^{-1/2})$ or $O(p^{-1})$ (see eqs. (4.5) and (4.25)).

Using the notation

$$\tilde{\xi}_j(\vec{\phi}) = \delta(\vec{\phi} - \vec{\phi}_j), \quad \tilde{\eta}_k(\vec{\psi}) = \delta(\vec{\psi} - \vec{\psi}_k), \quad (3.9)$$

we obtain

$$\begin{aligned} & \left\langle \exp \left(-i \sum_{j=1}^N \sum_{k=1}^M C_{jk} \vec{\phi}_j \cdot \vec{\psi}_k \right) \right\rangle \\ & \sim \exp \left\{ pN \sum_{j=1}^N \sum_{k=1}^M P_j Q_k \int d\vec{\phi} \int d\vec{\psi} \tilde{\xi}_j(\vec{\phi}) \tilde{\eta}_k(\vec{\psi}) \left(e^{-i\vec{\phi} \cdot \vec{\psi}} - 1 \right) \right\}, \end{aligned} \quad (3.10)$$

so that we find

$$\begin{aligned} \langle Z^n \rangle & \sim \int \prod_{j=1}^N d\vec{\phi}_j \int \prod_{k=1}^M d\vec{\psi}_k \\ & \times \exp \left\{ \frac{i}{2} \mu \sum_{j=1}^N \int d\vec{\phi} \tilde{\xi}_j(\vec{\phi}) \vec{\phi}^2 + \frac{i}{2} \mu \sum_{k=1}^M \int d\vec{\psi} \tilde{\eta}_k(\vec{\psi}) \vec{\psi}^2 \right\} \\ & \times \exp \left[pN \sum_{j=1}^N \sum_{k=1}^M P_j Q_k \int d\vec{\phi} \int d\vec{\psi} \tilde{\xi}_j(\vec{\phi}) \tilde{\eta}_k(\vec{\psi}) \left\{ f(\vec{\phi}, \vec{\psi}) - 1 \right\} \right] \\ & = \int \prod_{j=1}^N d\vec{\phi}_j \int \prod_{k=1}^M d\vec{\psi}_k \int \prod_{j=1}^N \mathcal{D}\xi_j(\vec{\phi}) \int \prod_{k=1}^M \mathcal{D}\eta_k(\vec{\psi}) \\ & \times \prod_{j=1}^N \prod_{\vec{\phi}} \delta(\xi_j(\vec{\phi}) - \tilde{\xi}_j(\vec{\phi})) \prod_{k=1}^M \prod_{\vec{\psi}} \delta(\eta_k(\vec{\psi}) - \tilde{\eta}_k(\vec{\psi})) e^{S_1 + S_2}. \end{aligned} \quad (3.11)$$

Here

$$f(\vec{\phi}, \vec{\psi}) = e^{-i\vec{\phi} \cdot \vec{\psi}}, \quad (3.12)$$

$$S_1 = \frac{i}{2} \mu \sum_{j=1}^N \int d\vec{\phi} \xi_j(\vec{\phi}) \vec{\phi}^2 + \frac{i}{2} \mu \sum_{k=1}^M \int d\vec{\psi} \eta_k(\vec{\psi}) \vec{\psi}^2 \quad (3.13)$$

and

$$S_2 = pN \sum_{j=1}^N \sum_{k=1}^M P_j Q_k \int d\vec{\phi} \int d\vec{\psi} \xi_j(\vec{\phi}) \eta_k(\vec{\psi}) \{f(\vec{\phi}, \vec{\theta}) - 1\}. \quad (3.14)$$

The functional integrations are taken over the auxiliary functions $\xi_j(\vec{\phi})$ and $\eta_k(\vec{\psi})$ satisfying

$$\int d\vec{\phi} \xi_j(\vec{\phi}) = \int d\vec{\psi} \eta_k(\vec{\psi}) = 1. \quad (3.15)$$

If J is the Laplacian matrix \mathcal{L} , we can similarly derive the same formula (3.11) for $\langle Z^n \rangle$, except that

$$f(\vec{\phi}, \vec{\psi}) = e^{-\frac{i}{2}(\vec{\phi} - \vec{\psi})^2}. \quad (3.16)$$

It can readily be seen that

$$\begin{aligned} & \int \prod_{j=1}^N d\vec{\phi}_j \prod_{j=1}^N \prod_{\vec{\phi}} \delta(\xi_j(\vec{\phi}) - \tilde{\xi}_j(\vec{\phi})) \\ &= \int \prod_{j=1}^N d\vec{\phi}_j \int \prod_{j=1}^N \mathcal{D}a_j(\vec{\phi}) \exp \left[2\pi i \sum_{j=1}^N \int d\vec{\phi} a_j(\vec{\phi}) \{ \xi_j(\vec{\phi}) - \tilde{\xi}_j(\vec{\phi}) \} \right] \\ &= \int \prod_{j=1}^N \mathcal{D}a_j(\vec{\phi}) \exp \left[\sum_{j=1}^N \left\{ 2\pi i \int d\vec{\phi} a_j(\vec{\phi}) \xi_j(\vec{\phi}) - F_j \right\} \right], \end{aligned} \quad (3.17)$$

where

$$\begin{aligned} F_j &= -\ln \int d\vec{\phi}_j \exp \left\{ -2\pi i \int d\vec{\phi} a_j(\vec{\phi}) \tilde{\xi}_j(\vec{\phi}) \right\} \\ &= -\ln \int d\vec{\phi}_j \exp \left\{ -2\pi i a_j(\vec{\phi}_j) \right\}. \end{aligned} \quad (3.18)$$

In the limit $N \rightarrow \infty$, the dominant contribution comes from the stationary point satisfying

$$\frac{\delta}{\delta a_j(\vec{\phi})} \left\{ 2\pi i \int d\vec{\phi} a_j(\vec{\phi}) \xi_j(\vec{\phi}) - F_j \right\} = 2\pi i \xi_j(\vec{\phi}) - 2\pi i e^{-2\pi i a_j(\vec{\phi}) + F_j} = 0, \quad (3.19)$$

which means

$$-\int d\vec{\phi} \xi_j(\vec{\phi}) \ln \xi_j(\vec{\phi}) = 2\pi i \int d\vec{\phi} a_j(\vec{\phi}) \xi_j(\vec{\phi}) - F_j. \quad (3.20)$$

Therefore we find an asymptotic estimate

$$\int \prod_{j=1}^N d\vec{\phi}_j \prod_{j=1}^N \prod_{\vec{\phi}} \delta(\xi_j(\vec{\phi}) - \tilde{\xi}_j(\vec{\phi})) \sim \exp \left\{ - \sum_{j=1}^N \int d\vec{\phi} \xi_j(\vec{\phi}) \ln \xi_j(\vec{\phi}) \right\}. \quad (3.21)$$

One can similarly derive another estimate

$$\int \prod_{k=1}^M d\vec{\psi}_k \prod_{k=1}^M \prod_{\vec{\psi}} \delta(\eta_k(\vec{\psi}) - \tilde{\eta}_k(\vec{\psi})) \sim \exp \left\{ - \sum_{k=1}^M \int d\vec{\psi} \eta_k(\vec{\psi}) \ln \eta_k(\vec{\psi}) \right\} \quad (3.22)$$

in the limit $M \rightarrow \infty$. Then we arrive at

$$\langle Z^n \rangle \sim \int \prod_{j=1}^N \mathcal{D}\xi_j(\vec{\phi}) \prod_{k=1}^M \mathcal{D}\eta_k(\vec{\psi}) e^{S_0 + S_1 + S_2}, \quad (3.23)$$

where

$$S_0 = - \sum_{j=1}^N \int d\vec{\phi} \xi_j(\vec{\phi}) \ln \xi_j(\vec{\phi}) - \sum_{k=1}^M \int d\vec{\psi} \eta_k(\vec{\psi}) \ln \eta_k(\vec{\psi}). \quad (3.24)$$

The functional integrations over $\xi_j(\vec{\phi})$ and $\eta_k(\vec{\psi})$ are dominated by the stationary point satisfying

$$\delta \left\{ S_0 + S_1 + S_2 + \sum_{j=1}^N \theta_j \left(\int d\vec{\phi} \xi_j(\vec{\phi}) - 1 \right) + \sum_{k=1}^M \omega_k \left(\int d\vec{\psi} \eta_k(\vec{\psi}) - 1 \right) \right\} = 0, \quad (3.25)$$

where θ_j and ω_k are the Lagrange multipliers. It follows from this equation that

$$\begin{aligned} \xi_j(\vec{\phi}) &= \Theta_j \exp \left[\frac{i}{2} \mu \vec{\phi}^2 + p N P_j \sum_{k=1}^M Q_k \int d\vec{\psi} \eta_k(\vec{\psi}) \{ f(\vec{\phi}, \vec{\psi}) - 1 \} \right], \\ \eta_k(\vec{\psi}) &= \Omega_j \exp \left[\frac{i}{2} \mu \vec{\psi}^2 + p N Q_k \sum_{j=1}^N P_j \int d\vec{\phi} \xi_j(\vec{\phi}) \{ f(\vec{\phi}, \vec{\psi}) - 1 \} \right], \end{aligned} \quad (3.26)$$

where Θ_j and Ω_k are normalisation constants.

4 The Limit of Large Mean Degree

In the limit of large mean degree $p \rightarrow \infty$, the variational equations (3.26) are satisfied by the Gaussian ansatz

$$\xi_j(\vec{\phi}) = \frac{1}{(2\pi i\sigma_j)^{n/2}} \exp\left(-\frac{\vec{\phi}^2}{2i\sigma_j}\right), \quad \eta_k(\vec{\psi}) = \frac{1}{(2\pi i\tau_k)^{n/2}} \exp\left(-\frac{\vec{\psi}^2}{2i\tau_k}\right), \quad (4.1)$$

as shown below. Here $\text{Im}\sigma_j \leq 0$ and $\text{Im}\tau_k \leq 0$. Putting the Gaussian ansatz (4.1) into (3.26), we find

$$\begin{aligned} \xi_j(\vec{\phi}) &= \Theta_j \exp\left[\frac{i}{2}\mu\vec{\phi}^2 + pNP_j \sum_{k=1}^M Q_k \{h_k(\vec{\phi}^2) - 1\}\right], \\ \eta_k(\vec{\psi}) &= \Omega_j \exp\left[\frac{i}{2}\mu\vec{\psi}^2 + pNQ_k \sum_{j=1}^N P_j \{h_j(\vec{\psi}^2) - 1\}\right], \end{aligned} \quad (4.2)$$

where

$$h_k(\vec{\phi}) = \begin{cases} \exp\left(-\frac{i\tau_k}{2}\vec{\phi}^2\right), & \text{If } J \text{ is } \mathcal{A}, \\ \exp\left(-\frac{i}{2(1-\tau_k)}\vec{\phi}^2\right), & \text{If } J \text{ is } \mathcal{L} \end{cases} \quad (4.3)$$

and

$$h_j(\vec{\psi}) = \begin{cases} \exp\left(-\frac{i\sigma_j}{2}\vec{\psi}^2\right), & \text{If } J \text{ is } \mathcal{A}, \\ \exp\left(-\frac{i}{2(1-\sigma_j)}\vec{\psi}^2\right), & \text{If } J \text{ is } \mathcal{L}. \end{cases} \quad (4.4)$$

Let us first consider the adjacency matrix \mathcal{A} . We are in a position to take the limit $p \rightarrow \infty$ with the scalings

$$\mu = O(p^{1/2}), \quad \vec{\phi}^2 = O(p^{-1/2}), \quad \sigma_j = O(p^{-1/2}), \quad \vec{\psi}^2 = O(p^{-1/2}), \quad \tau_k = O(p^{-1/2}). \quad (4.5)$$

Then we obtain

$$\mu - \frac{1}{\sigma_j} - pNP_j \sum_{k=1}^M Q_k \tau_k = 0, \quad \mu - \frac{1}{\tau_k} - pNQ_k \sum_{j=1}^N P_j \sigma_j = 0. \quad (4.6)$$

The variational equations (3.26) are satisfied by the Gaussian ansatz (4.1), if σ_j and τ_k are determined by these equations.

In order to analytically treat (4.6), we define the scaling variables

$$E = \mu/\sqrt{p}, \quad s(x) = \sqrt{p} \sigma_j, \quad x = j/N, \quad t(y) = \sqrt{p} \tau_k, \quad y = k/M. \quad (4.7)$$

Then it is straightforward to find

$$\frac{x^\alpha}{s(x)} = Ex^\alpha - (1-\alpha)(1-\beta) \int_0^1 y^{-\beta} t(y) dy \quad (4.8)$$

and

$$\frac{y^\beta}{t(y)} = Ey^\beta - c(1-\alpha)(1-\beta) \int_0^1 x^{-\alpha} s(x) dx \quad (4.9)$$

in the limit (2.1). Using the notations

$$S = c(1-\alpha)(1-\beta) \int_0^1 x^{-\alpha} s(x) dx, \quad T = (1-\alpha)(1-\beta) \int_0^1 y^{-\beta} t(y) dy, \quad (4.10)$$

we obtain

$$\frac{S}{c(1-\alpha)(1-\beta)} = \int_0^1 \frac{1}{Ex^\alpha - T} dx, \quad \frac{T}{(1-\alpha)(1-\beta)} = \int_0^1 \frac{1}{Ey^\beta - S} dy. \quad (4.11)$$

In order to evaluate the behaviour of S and T in the tail region $E \rightarrow \infty$, we write

$$S = E(s^{(R)} - is^{(I)}), \quad T = E(t^{(R)} - it^{(I)}) \quad (4.12)$$

with real $s^{(R)}$, $s^{(I)}$, $t^{(R)}$ and $t^{(I)}$. Then it can be seen that

$$\begin{aligned} \frac{E(s^{(R)} - is^{(I)})}{c(1-\alpha)(1-\beta)} &= \frac{1}{E} \int_0^1 \frac{1}{x^\alpha - t^{(R)} + it^{(I)}} dx = \frac{1}{\alpha E} \int_0^1 \frac{s^{(1-\alpha)/\alpha}}{s - t^{(R)} + it^{(I)}} ds \\ &= \frac{1}{\alpha E} \int_0^1 \frac{s^{(1-\alpha)/\alpha} (s - t^{(R)})}{(s - t^{(R)})^2 + (t^{(I)})^2} ds - \frac{i}{\alpha E} \int_0^1 \frac{s^{(1-\alpha)/\alpha} t^{(I)}}{(s - t^{(R)})^2 + (t^{(I)})^2} ds. \end{aligned} \quad (4.13)$$

Let us employ an asymptotic formula[6]

$$\frac{\epsilon}{(u-a)^2 + \epsilon^2} \sim \pi \delta(u-a), \quad \epsilon \downarrow 0 \quad (4.14)$$

and obtain an estimate

$$\begin{aligned} \frac{E(s^{(R)} + is^{(I)})}{c(1-\alpha)(1-\beta)} &\sim \frac{1}{\alpha E} \int_0^1 s^{(1/\alpha)-2} ds - \frac{i\pi}{\alpha E} \int_0^1 s^{(1-\alpha)/\alpha} \delta(s - t^{(R)}) ds \\ &= \frac{1}{E(1-\alpha)} - \frac{i\pi}{\alpha E} (t^{(R)})^{(1-\alpha)/\alpha}, \quad E \rightarrow \infty, \end{aligned} \quad (4.15)$$

so that

$$s^{(R)} \sim \frac{c(1-\beta)}{E^2}, \quad s^{(I)} \sim \frac{c(1-\alpha)(1-\beta)\pi}{\alpha E^2} (t^{(R)})^{(1-\alpha)/\alpha}. \quad (4.16)$$

One can similarly derive

$$\begin{aligned} \frac{E(t^{(R)} - it^{(I)})}{(1-\alpha)(1-\beta)} &= \frac{1}{E} \int_0^1 \frac{1}{y^\beta - s^{(R)} + is^{(I)}} dx = \frac{1}{\beta E} \int_0^1 \frac{t^{(1-\beta)/\beta}}{t - s^{(R)} + is^{(I)}} dt \\ &\sim \frac{1}{E(1-\beta)} - \frac{i\pi}{\beta E} (s^{(R)})^{(1-\beta)/\beta}, \quad E \rightarrow \infty, \end{aligned} \quad (4.17)$$

so that

$$t^{(R)} \sim \frac{1-\alpha}{E^2}, \quad t^{(I)} \sim \frac{(1-\alpha)(1-\beta)\pi}{\beta E^2} (s^{(R)})^{(1-\beta)/\beta}. \quad (4.18)$$

It follows from (4.16) and (4.18) that

$$s^{(I)} \sim \frac{1-\beta}{\alpha} \pi c \left(\frac{1-\alpha}{E^2} \right)^{1/\alpha}, \quad t^{(I)} \sim \frac{1-\alpha}{\beta} \pi c^{(1-\beta)/\beta} \left(\frac{1-\beta}{E^2} \right)^{1/\beta}. \quad (4.19)$$

Now we can evaluate the asymptotic behaviour of the spectral density $\rho(\mu)$ in the tail region $E \rightarrow \infty$. Eqs. (2.28) and (3.23) can be utilised as

$$\begin{aligned} \rho(\mu) &= \lim_{n \rightarrow 0} \frac{2}{(N+M)n\pi} \text{Im} \frac{\partial}{\partial \mu} (S_0 + S_1 + S_2) \\ &= \lim_{n \rightarrow 0} \frac{1}{(N+M)n\pi} \text{Re} \left(\sum_{j=1}^N \int d\vec{\phi} \, \xi_j(\vec{\phi}) \vec{\phi}^2 + \sum_{k=1}^M \int d\vec{\psi} \, \eta_k(\vec{\psi}) \vec{\psi}^2 \right) \\ &= -\frac{1}{(N+M)\pi} \text{Im} \left(\sum_{j=1}^N \sigma_j + \sum_{k=1}^M \tau_k \right) \\ &\sim -\frac{1}{\sqrt{p}(1+c)\pi} \text{Im} \left(c \int_0^1 s(x) dx + \int_0^1 t(y) dy \right) \end{aligned} \quad (4.20)$$

in the limit (2.1). Here

$$\begin{aligned} \int_0^1 s(x) dx &= \int_0^1 \frac{1}{E - T x^{-\alpha}} dx = \frac{1}{E} + \frac{T}{E} \int_0^1 \frac{1}{E x^\alpha - T} dx \\ &= \frac{1}{E} + \frac{ST}{cE(1-\alpha)(1-\beta)} \end{aligned} \quad (4.21)$$

and we can similarly obtain

$$\int_0^1 t(y) dy = \frac{1}{E} + \frac{ST}{E(1-\alpha)(1-\beta)}. \quad (4.22)$$

Then it can be seen from (4.16), (4.18) and (4.19) that

$$\begin{aligned} \rho(\mu) &\sim -\frac{2}{\sqrt{p}E(1+c)\pi(1-\alpha)(1-\beta)} \text{Im}(ST) \\ &= \frac{2E}{\sqrt{p}(1+c)\pi(1-\alpha)(1-\beta)} \left\{ s^{(R)}t^{(I)} + s^{(I)}t^{(R)} \right\} \\ &\sim \frac{2}{\sqrt{p}(1+c)} \left\{ \frac{c^{1/\beta}(1-\beta)^{1/\beta}}{\beta E^{(2/\beta)+1}} + \frac{c(1-\alpha)^{1/\alpha}}{\alpha E^{(2/\alpha)+1}} \right\}. \end{aligned} \quad (4.23)$$

This gives the asymptotic spectral density of the adjacency matrix \mathcal{A} in the tail region $E \rightarrow \infty$. The exponent γ of the spectral density defined as

$$\rho(\mu) \propto \mu^{-\gamma}, \quad \mu \rightarrow \infty \quad (4.24)$$

is $(2/\alpha) + 1$ if $\alpha \geq \beta$, and is $(2/\beta) + 1$ if $\beta \geq \alpha$. Thus γ is associated with $\lambda_A = (1/\alpha) + 1$ and $\lambda_B = (1/\beta) + 1$ as $\gamma = 2 \min(\lambda_A, \lambda_B) - 1$.

We next compute the spectral density of the Laplacian matrix \mathcal{L} . Using the scalings

$$\mu = O(p), \quad \vec{\phi}^2 = O(p^{-1}), \quad \sigma_j = O(p^{-1}), \quad \vec{\psi}^2 = O(p^{-1}), \quad \tau_k = O(p^{-1}) \quad (4.25)$$

and taking the limit $p \rightarrow \infty$, we find

$$\mu - \frac{1}{\sigma_j} - pNP_j = 0, \quad \mu - \frac{1}{\tau_k} - pNQ_k = 0, \quad (4.26)$$

so that

$$\begin{aligned} \text{Im}\sigma_j &= \text{Im} \frac{1}{\mu + i\epsilon - pNP_j} = -\pi\delta(\mu - pNP_j), \\ \text{Im}\tau_k &= \text{Im} \frac{1}{\mu + i\epsilon - pNQ_k} = -\pi\delta(\mu - pNQ_k), \end{aligned} \quad (4.27)$$

where ϵ is an infinitesimal positive number. Then it follows in the limit (2.1) that

$$\begin{aligned} \text{Im} \sum_{j=1}^N \sigma_j &\sim -N\pi \int_0^1 dx \delta(\mu - p(1-\alpha)x^{-\alpha}) \\ &= -N\pi \frac{\{p(1-\alpha)\}^{1/\alpha}}{\alpha} \frac{1}{\mu^{(1/\alpha)+1}} H\{\mu - p(1-\alpha)\} \end{aligned} \quad (4.28)$$

and

$$\begin{aligned} \text{Im} \sum_{k=1}^M \tau_k &\sim -M\pi \int_0^1 dy \delta(\mu - pc(1-\beta)y^{-\beta}) \\ &= -M\pi \frac{\{pc(1-\beta)\}^{1/\beta}}{\beta} \frac{1}{\mu^{(1/\beta)+1}} H\{\mu - pc(1-\beta)\}, \end{aligned} \quad (4.29)$$

where

$$H(x) = \begin{cases} 0, & x < 0, \\ 1, & x > 0. \end{cases} \quad (4.30)$$

Therefore we arrive at

$$\begin{aligned} \rho(\mu) &= -\frac{1}{(N+M)\pi} \text{Im} \left(\sum_{j=1}^N \sigma_j + \sum_{k=1}^M \tau_k \right) \\ &\sim \frac{c\{p(1-\alpha)\}^{1/\alpha}}{(1+c)\alpha} \frac{1}{\mu^{(1/\alpha)+1}} H\{\mu - p(1-\alpha)\} \\ &\quad + \frac{\{pc(1-\beta)\}^{1/\beta}}{(1+c)\beta} \frac{1}{\mu^{(1/\beta)+1}} H\{\mu - pc(1-\beta)\}. \end{aligned} \quad (4.31)$$

This gives the asymptotic spectral density of the Laplacian matrix \mathcal{L} in the region $\mu = O(p)$. The exponent γ_L of the spectral density $\rho(\mu) \propto \mu^{-\gamma_L}$ ($\mu \rightarrow \infty$) is $(1/\alpha) + 1$ if $\alpha \geq \beta$, and is $(1/\beta) + 1$ if $\beta \geq \alpha$. Thus γ_L is associated with λ_A and λ_B as $\gamma_L = \min(\lambda_A, \lambda_B)$.

5 Effective Medium Approximation

In the previous section we have dealt with the spectral density in the limit $p \rightarrow \infty$. The calculation of the spectral density with a finite mean degree p is a much more involved problem, for which sophisticated numerical schemes have been proposed[9, 10]. In this section we briefly discuss a simple approximation method (effective medium approximation) for that problem[8, 11, 12, 13, 14, 15]. In this approximation, we put the Gaussian ansatz (4.1) into the formulas (3.13), (3.14) and (3.24), and solve the stationary point equations

$$\frac{\partial}{\partial \sigma_j} (S_0 + S_1 + S_2) = 0 \quad (5.1)$$

and

$$\frac{\partial}{\partial \tau_k}(S_0 + S_1 + S_2) = 0. \quad (5.2)$$

In the case of the adjacency matrix \mathcal{A} , the above procedure results in the effective medium approximation (EMA) equations

$$\begin{aligned} \mu - \frac{1}{\sigma_j} - pNP_j \sum_{k=1}^M \frac{Q_k \tau_k}{1 - \sigma_j \tau_k} &= 0, \\ \mu - \frac{1}{\tau_k} - pNQ_k \sum_{j=1}^N \frac{P_j \sigma_j}{1 - \sigma_j \tau_k} &= 0. \end{aligned} \quad (5.3)$$

As for scale free networks with a single species of nodes, Nagao and Rodgers calculated the $1/p$ expansion of the spectral density by using the corresponding EMA equation[15]. A similar analytical treatment could also be possible in the present case. Here, however only a result of a numerical iteration of (5.3) is shown in Figure 1 as the EMA spectral density. It is compared with the spectral density of positive eigenvalues calculated by numerical diagonalisations of numerically generated adjacency matrices (averaged over 100 samples). The agreement seems fairly good.

In the limit $\alpha, \beta \rightarrow 0$, we obtain the adjacency matrix of a classical random graph with two species, where the connections are made only between the nodes of different species. In that case σ_j and τ_k can be written as σ and τ , respectively, because they depend on neither j nor k . The EMA equations become a cubic equation for σ

$$c\sigma^3 + \frac{c(p-2)+1}{\mu}\sigma^2 - \frac{\mu^2 + (c-1)(p-1)}{\mu^2}\sigma + \frac{1}{\mu} = 0 \quad (5.4)$$

and

$$\tau = \frac{c(\mu\sigma - 1) + 1}{\mu}. \quad (5.5)$$

These equations are equivalent to Nagao and Tanaka's SEMA (symmetric EMA) equations concerning the spectral density of sparse correlation matrices[14], and can be analysed in the same way.

We can similarly derive the EMA equations for the Laplacian matrix \mathcal{L} as

$$\mu - \frac{1}{\sigma_j} - pNP_j \sum_{k=1}^M \frac{Q_k}{1 - \sigma_j - \tau_k} = 0,$$

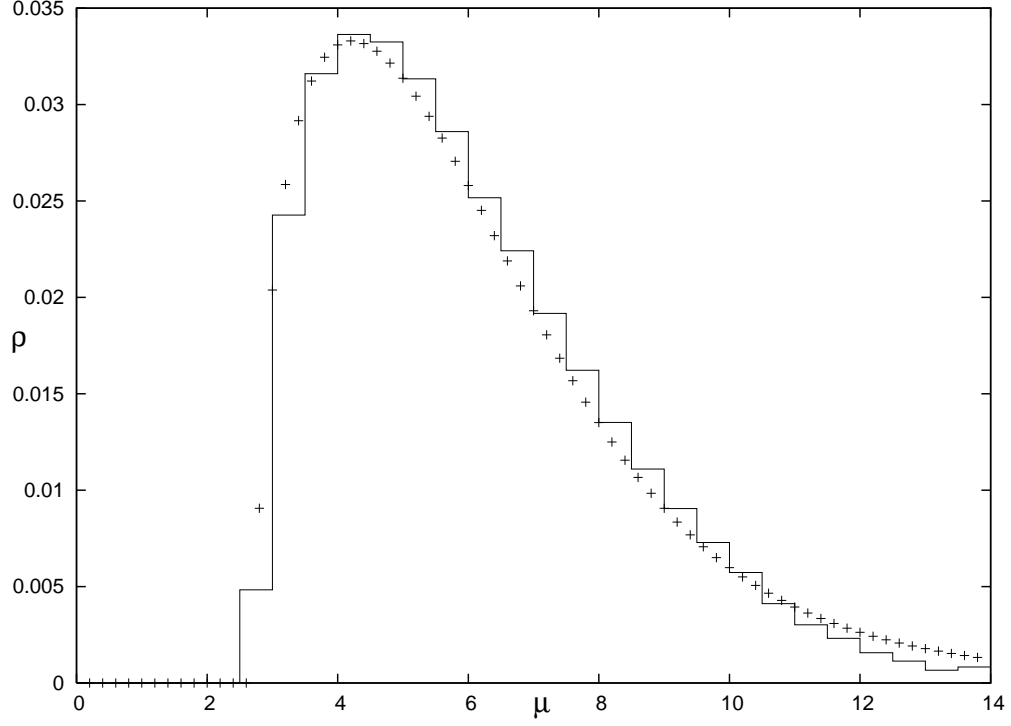


Figure 1: The EMA spectral density (+) and the spectral density of numerically generated adjacency matrices (histogram). The parameters are $N = 1000$, $M = 200$, $\alpha = \beta = 1/2$ and $p = 10$.

$$\mu - \frac{1}{\tau_k} - pNQ_k \sum_{j=1}^N \frac{P_j}{1 - \sigma_j - \tau_k} = 0. \quad (5.6)$$

In the limit $\alpha, \beta \rightarrow 0$, σ_j and τ_k can again be reduced to σ and τ , respectively. Then we find a cubic equation for σ

$$(1 - c)\sigma^3 + \frac{2c + (p - \mu)(1 - c)}{\mu}\sigma^2 + \frac{\mu - 1 + c(p - 2\mu - 1)}{\mu^2}\sigma + \frac{c}{\mu^2} = 0 \quad (5.7)$$

and

$$\tau = \frac{\sigma}{(1 - c)\mu\sigma + c}. \quad (5.8)$$

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Appendix

In this Appendix, we derive an asymptotic relation

$$\ln \left\langle \exp \left(-i \sum_{j=1}^N \sum_{k=1}^M C_{jk} t_{jk} \right) \right\rangle \sim pN \sum_{j=1}^N \sum_{k=1}^M P_j Q_k (e^{-it_{jk}} - 1), \quad (\text{A.1})$$

where t_{jk} is a parameter which is independent of N and M . We moreover assume that t_{jk} is in the neighbourhood of the origin so that $|S_{jk}| < 1$ holds for $S_{jk} = e^{-it_{jk}} - 1$. A similar argument for Goh, Kahng and Kim's model is found in [16].

The Taylor expansion of the logarithmic function gives

$$\begin{aligned} \ln \left\langle \exp \left(-i \sum_{j=1}^N \sum_{k=1}^M C_{jk} t_{jk} \right) \right\rangle &= \sum_{j=1}^N \sum_{k=1}^M \ln(1 + f_{jk} S_{jk}) \\ &= pN \sum_{j=1}^N \sum_{k=1}^M P_j Q_k S_{jk} + \sum_{j=1}^N \sum_{k=1}^M (f_{jk} - pN P_j Q_k) S_{jk} \\ &\quad + \sum_{\ell=2}^{\infty} \frac{(-1)^{\ell+1}}{\ell} \sum_{j=1}^N \sum_{k=1}^M (f_{jk} S_{jk})^{\ell}. \end{aligned} \quad (\text{A.2})$$

We show

$$\ln \left\langle \exp \left(-i \sum_{j=1}^N \sum_{k=1}^M C_{jk} t_{jk} \right) \right\rangle - pN \sum_{j=1}^N \sum_{k=1}^M P_j Q_k S_{jk} = o(N) \quad (\text{A.3})$$

in two steps.

Step 1

Let us first prove that

$$\sum_{j=1}^N \sum_{k=1}^M (f_{jk} - pN P_j Q_k) S_{jk} = o(N). \quad (\text{A.4})$$

We define

$$S_{\max} = \max_{jk} |S_{jk}| \quad (\text{A.5})$$

and

$$G_1(x) = x - 1 + e^{-x}. \quad (\text{A.6})$$

Then we see that

$$\begin{aligned} \left| \sum_{j=1}^N \sum_{k=1}^M (f_{jk} - pNP_j Q_k) S_{jk} \right| &\leq S_{\max} \sum_{j=1}^N \sum_{k=1}^M |f_{jk} - pNP_j Q_k| \\ &= S_{\max} \sum_{j=1}^N \sum_{k=1}^M G_1(pNP_j Q_k). \end{aligned} \quad (\text{A.7})$$

A monotonously decreasing continuous function $F(x)$ satisfies

$$\sum_{k=1}^M F(k) \leq \int_1^M F(y) \, dy + F(1), \quad (\text{A.8})$$

so that

$$\sum_{k=1}^M G_1(pNP_j Q_k) \leq \int_1^M G_1 \left(pNP_j \frac{y^{-\beta}}{\zeta_M(\beta)} \right) dy + G_1 \left(pNP_j \frac{1}{\zeta_M(\beta)} \right) \quad (\text{A.9})$$

with

$$\zeta_M(\beta) = \sum_{\ell=1}^M \ell^{-\beta}. \quad (\text{A.10})$$

Then one can again use (A.8) to obtain

$$\sum_{j=1}^N \sum_{k=1}^M G_1(pNP_j Q_k) \leq \sum_{\nu=1}^4 I_{\nu}, \quad (\text{A.11})$$

where

$$\begin{aligned} I_1 &= \int_1^N dx \int_1^M dy \, G_1 \left(pN \frac{x^{-\alpha}}{\zeta_N(\alpha)} \frac{y^{-\beta}}{\zeta_M(\beta)} \right), \\ I_2 &= \int_1^M dy \, G_1 \left(pN \frac{1}{\zeta_N(\alpha)} \frac{y^{-\beta}}{\zeta_M(\beta)} \right), \quad I_3 = \int_1^N dx \, G_1 \left(pN \frac{x^{-\alpha}}{\zeta_N(\alpha)} \frac{1}{\zeta_M(\beta)} \right), \\ I_4 &= G_1 \left(pN \frac{1}{\zeta_N(\alpha)} \frac{1}{\zeta_M(\beta)} \right). \end{aligned} \quad (\text{A.12})$$

Using the notations

$$\epsilon_1 = \frac{\sqrt{pN}}{\zeta_N(\alpha)} N^{-\alpha}, \quad \epsilon_2 = \frac{\sqrt{pN}}{\zeta_M(\beta)} M^{-\beta}, \quad (\text{A.13})$$

we see that

$$I_1 = \frac{MN\epsilon_1^{1/\alpha}\epsilon_2^{1/\beta}}{\alpha\beta} \int_{\epsilon_1}^{\epsilon_1 N^\alpha} du \int_{\epsilon_2}^{\epsilon_2 M^\beta} dv \frac{G_1(uv)}{u^{1+(1/\alpha)}v^{1+(1/\beta)}}. \quad (\text{A.14})$$

Then, using the inequality

$$G_1(x) \leq x^2/2, \quad x \geq 0, \quad (\text{A.15})$$

we find

$$\begin{aligned} I_1 &\leq \frac{MN\epsilon_1^{1/\alpha}\epsilon_2^{1/\beta}}{2\alpha\beta} \int_{\epsilon_1}^{\epsilon_1 N^\alpha} du \int_{\epsilon_2}^{\epsilon_2 M^\beta} dv u^{1-(1/\alpha)}v^{1-(1/\beta)} \\ &= O(N^{(2\alpha-1)}N^{(2\beta-1)}), \end{aligned} \quad (\text{A.16})$$

where

$$N^{(2\alpha-1)} \equiv \begin{cases} 1, & 0 < \alpha < 1/2, \\ \ln N, & \alpha = 1/2, \\ N^{2\alpha-1}, & 1/2 < \alpha < 1. \end{cases} \quad (\text{A.17})$$

In the case $\min(\alpha, \beta) > 1/2$, we similarly employ

$$G_1(x) \leq \begin{cases} x^2/2, & 0 \leq x \leq 1, \\ x, & x \geq 1 \end{cases} \quad (\text{A.18})$$

to obtain

$$I_1 = O(N^{(\alpha, \beta)}), \quad (\text{A.19})$$

where

$$N^{(\alpha, \beta)} = \begin{cases} N^{(\alpha+\beta-1)/\min(\alpha, \beta)}, & \alpha \neq \beta, \\ N^{(2\alpha-1)/\alpha} \ln N, & \alpha = \beta. \end{cases} \quad (\text{A.20})$$

Using the inequality (A.15), we can similarly derive the estimates

$$I_2 = O(N^{2\alpha-1}N^{(2\beta-1)}), \quad I_3 = O(N^{(2\alpha-1)}N^{2\beta-1}). \quad (\text{A.21})$$

If $\min(\alpha, \beta) > 1/2$, we utilise (A.18) to find

$$I_2 = O(N^{(\alpha+\beta-1)/\beta}), \quad I_3 = O(N^{(\alpha+\beta-1)/\alpha}). \quad (\text{A.22})$$

Moreover, one can readily see from the inequality $G_1(x) \leq x$ ($x \geq 0$) that

$$I_4 = O(N^{\alpha+\beta-1}). \quad (\text{A.23})$$

It follows from (A.16), (A.19), (A.21), (A.22) and (A.23) that

$$\sum_{j=1}^N \sum_{k=1}^M G_1(pNP_jQ_k) \leq \sum_{\nu=1}^4 I_\nu = o(N), \quad (\text{A.24})$$

which yields (A.4).

Step 2

We next prove

$$\sum_{\ell=2}^{\infty} \frac{(-1)^{\ell+1}}{\ell} \sum_{j=1}^N \sum_{k=1}^M (f_{jk}S_{jk})^\ell = o(N). \quad (\text{A.25})$$

Using

$$G_0(x) = 1 - e^{-x}, \quad (\text{A.26})$$

we see that

$$\begin{aligned} \left| \sum_{\ell=2}^{\infty} \frac{(-1)^{\ell+1}}{\ell} \sum_{j=1}^N \sum_{k=1}^M (f_{jk}S_{jk})^\ell \right| &\leq \sum_{\ell=2}^{\infty} \frac{S_{\max}^\ell}{\ell} \sum_{j=1}^N \sum_{k=1}^M f_{jk}^\ell \\ &= \sum_{\ell=2}^{\infty} \frac{S_{\max}^\ell}{\ell} \sum_{j=1}^N \sum_{k=1}^M G_0(pNP_jQ_k)^\ell. \end{aligned} \quad (\text{A.27})$$

We can again employ (A.8) to obtain

$$\sum_{j=1}^N \sum_{k=1}^M G_0(pNP_jQ_k)^\ell \leq \sum_{\nu=1}^4 J_\nu, \quad (\text{A.28})$$

where

$$\begin{aligned} J_1 &= \int_1^N dx \int_1^M dy G_0 \left(pN \frac{x^{-\alpha}}{\zeta_N(\alpha)} \frac{y^{-\beta}}{\zeta_M(\beta)} \right)^\ell, \\ J_2 &= \int_1^M dy G_0 \left(pN \frac{1}{\zeta_N(\alpha)} \frac{y^{-\beta}}{\zeta_M(\beta)} \right)^\ell, \quad J_3 = \int_1^N dx G_0 \left(pN \frac{x^{-\alpha}}{\zeta_N(\alpha)} \frac{1}{\zeta_M(\beta)} \right)^\ell, \\ J_4 &= G_0 \left(pN \frac{1}{\zeta_N(\alpha)} \frac{1}{\zeta_M(\beta)} \right)^\ell. \end{aligned} \quad (\text{A.29})$$

Making use of the identity

$$G_0(x) \leq x, \quad x \geq 0, \quad (\text{A.30})$$

we find

$$J_1 = O\left(N^{\langle\alpha,\ell\rangle} N^{\langle\beta,\ell\rangle}\right), \quad (\text{A.31})$$

where

$$N^{\langle\alpha,\ell\rangle} = \begin{cases} N^{(2-\ell)/2}, & \ell < 1/\alpha, \\ N^{(2\alpha-1)/(2\alpha)} \ln N, & \ell = 1/\alpha, \\ N^{\ell(2\alpha-1)/2}, & \ell > 1/\alpha. \end{cases} \quad (\text{A.32})$$

In the case $\alpha + \beta > 1$, by means of

$$G_0(x) \leq \begin{cases} x, & 0 \leq x \leq 1, \\ 1, & x \geq 1, \end{cases} \quad (\text{A.33})$$

we obtain

$$J_1 = \begin{cases} O(N^{\langle\alpha,\beta\rangle}), & \alpha > 1/2, \beta > 1/2, \\ O(N^{2\alpha-1} \ln N), & \alpha > 1/2, \beta \leq 1/2, \ell = 1/\beta, \\ O(N^{2\alpha-1}), & \alpha > 1/2, \beta \leq 1/2, \ell \neq 1/\beta, \\ O(N^{2\beta-1} \ln N), & \alpha \leq 1/2, \beta > 1/2, \ell = 1/\alpha, \\ O(N^{2\beta-1}), & \alpha \leq 1/2, \beta > 1/2, \ell \neq 1/\alpha. \end{cases} \quad (\text{A.34})$$

Here the symbol $N^{\langle\alpha,\beta\rangle}$ is defined in (A.20). The inequality (A.33) similarly gives the estimates

$$\begin{aligned} J_2 &= \begin{cases} O(N^{1+\ell(\alpha-1)}), & \ell < 1/\beta, \\ O(N^{(\alpha+\beta-1)/\beta} \ln N), & \ell = 1/\beta, \\ O(N^{(\alpha+\beta-1)/\beta}), & \ell > 1/\beta, \end{cases} \\ J_3 &= \begin{cases} O(N^{1+\ell(\beta-1)}), & \ell < 1/\alpha, \\ O(N^{(\alpha+\beta-1)/\alpha} \ln N), & \ell = 1/\alpha, \\ O(N^{(\alpha+\beta-1)/\alpha}), & \ell > 1/\alpha. \end{cases} \end{aligned} \quad (\text{A.35})$$

Moreover it is evident from the inequality $G_0(x) \leq 1$ ($x \geq 0$) that

$$J_4 = O(1). \quad (\text{A.36})$$

Now we can easily see from (A.31), (A.34), (A.35) and (A.36) that

$$\sum_{j=1}^N \sum_{k=1}^M G_0(pNP_j Q_k)^\ell \leq \sum_{\nu=1}^4 J_\nu = o(N) \quad (\text{A.37})$$

for any $\ell \geq 2$. This relation results in the asymptotic estimate (A.25).